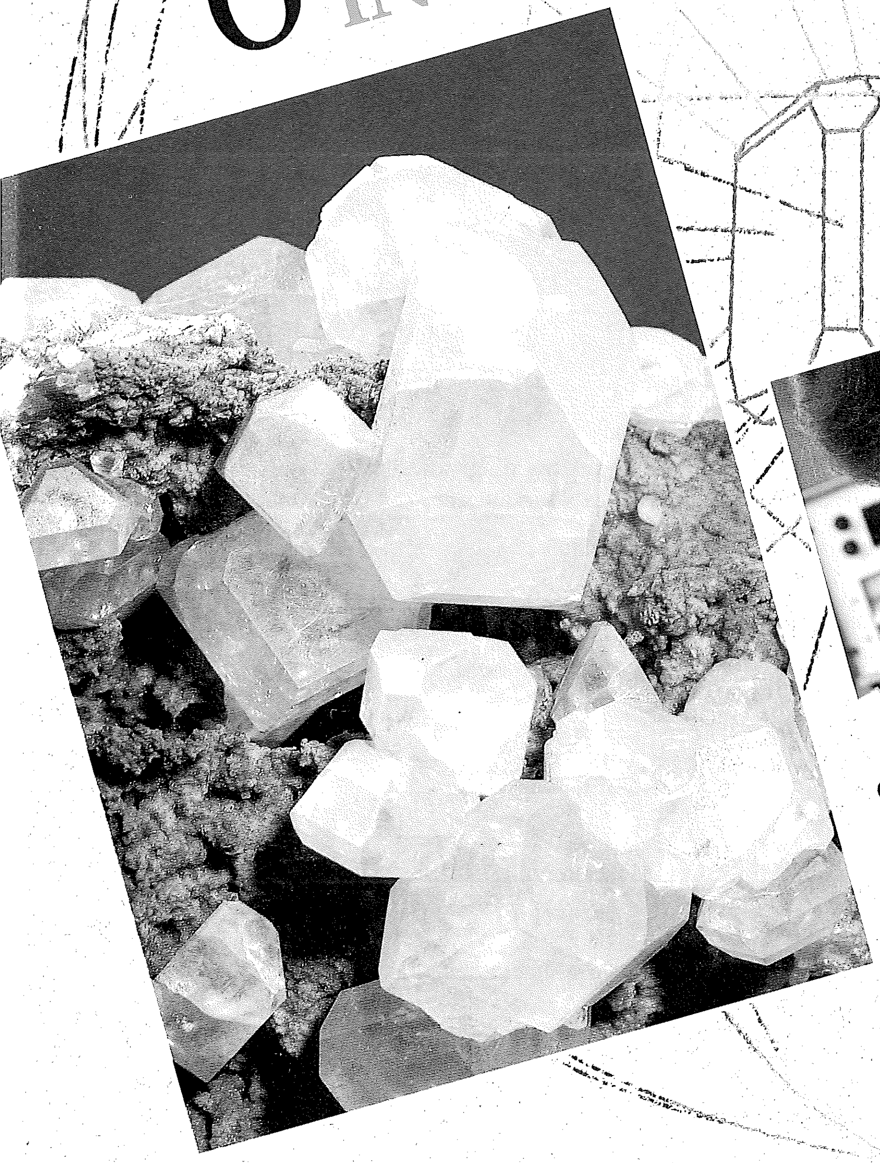
6 INEQUALITIES IN GEOMETRY



Crystallographers can easily identify sulfur crystals by the characteristic faces and angles produced by the symmetrical arrangement of the atoms.

Inequalities and Indirect Proof

Objectives

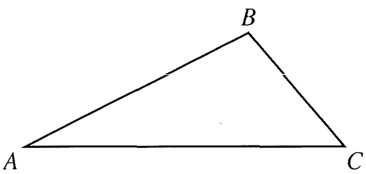
1. Apply properties of inequality to positive numbers, lengths of segments, and measures of angles.
2. State the contrapositive and inverse of an if-then statement.
3. Understand the relationship between logically equivalent statements.
4. Draw correct conclusions from given statements.
5. Write indirect proofs in paragraph form.

6-1 *Inequalities*

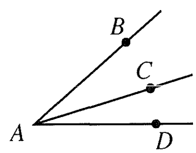
Our geometry up until now has emphasized congruent segments and angles, and the triangles and polygons they form. To deal with segments whose lengths are equal and angles whose measures are equal, you have used properties of equality taken from algebra. They are stated on page 37. In this chapter you will work with segments having unequal lengths and angles having unequal measures. You will use properties of inequality taken from algebra.

Example 1 Complete each conclusion by inserting one of the symbols $<$, $=$, or $>$.

a. Given: $AC > AB$; $AB > BC$
Conclusion: $AC \underline{\ ?\ } BC$



b. Given: $m\angle BAC + m\angle CAD = m\angle BAD$
Conclusion: $m\angle BAD \underline{\ ?\ } m\angle BAC$;
$m\angle BAD \underline{\ ?\ } m\angle CAD$



Solution **a.** $AC > BC$ (Equivalently, $BC < AC$.)

b. $m\angle BAD > m\angle BAC$ (Equivalently, $m\angle BAC < m\angle BAD$.)
$m\angle BAD > m\angle CAD$ (Equivalently, $m\angle CAD < m\angle BAD$.)

The properties of inequality you will use most often in geometry are stated on the following page. When you use any one of them in a proof, you can write as your reason *A Prop. of Ineq.* Can you see which properties were used in Example 1?

Properties of Inequality

If $a > b$ and $c \geq d$, then $a + c > b + d$.

If $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

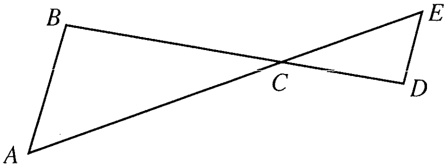
If $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

If $a > b$ and $b > c$, then $a > c$.
If $a = b + c$ and $c > 0$, then $a > b$.

Example 2

Given: $AC > BC$; $CE > CD$

Prove: $AE > BD$



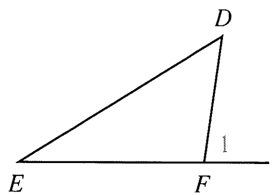
Proof:

Statements	Reasons
1. $AC > BC$; $CE > CD$	1. Given
2. $AC + CE > BC + CD$	2. A Prop. of Ineq.
3. $AC + CE = AE$; $BC + CD = BD$	3. Segment Addition Postulate
4. $AE > BD$	4. Substitution Prop.

Example 3

Given: $\angle 1$ is an exterior angle of $\triangle DEF$.

Prove: $m\angle 1 > m\angle D$;
$m\angle 1 > m\angle E$



Proof:

Statements	Reasons
1. $m\angle 1 = m\angle D + m\angle E$	1. The measure of an ext. $\angle$ of a $\triangle$ equals the sum of the measures of the two remote int. $\angle$s.
2. $m\angle 1 > m\angle D$; $m\angle 1 > m\angle E$	2. A Prop. of Ineq.

Example 3 above proves the following theorem.

Theorem 6-1 *The Exterior Angle Inequality Theorem*

The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

Classroom Exercises

Classify each conditional as true or false.

1. If $3a > 9$, then $a > 27$.

2. If $4b > 20$, then $b > 5$.

3. If $x > 4$, then $x + 1 > 5$.

4. If $x + 1 > 5$, then $x > 4$.

5. If $c - 5 > 45$, then $c > 48$.

6. If $a + b = n$ and $c > b$, then $a + c > n$.

7. If $y > 18$, then $y > 20$.

8. If $y > 20$, then $y > 18$.

9. If $a > 5$ and $5 > b$, then $a > b$.

10. If $d > e$ and $f > e$, then $d > f$.

11. If $g > h$ and $j = h$, then $g > j$.

12. If $p = q + 6$, then $p > q$.

13. If $c > d$ and $e = f$, then $c + e = d + f$.

14. If $g > h$ and $i > j$, then $g + h > i + j$.

15. If $k > l$ and $m > n$, then $k + m > l + n$.

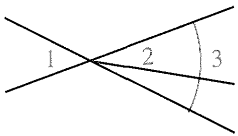
16. If $a > b$, then $100 - a > 100 - b$.

Complete each statement by writing <, =, or >.

17.

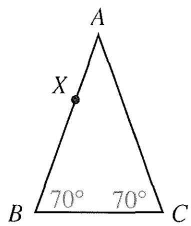
a. XZ _?_ $XY + YZ$
b. XZ _?_ XY
c. XZ _?_ YZ

18.



a. $m\angle 1$ _?_ $m\angle 3$
b. $m\angle 2$ _?_ $m\angle 3$
c. $m\angle 1$ _?_ $m\angle 2$

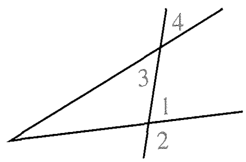
19.



a. AB _?_ AC
b. AB _?_ $AX + XB$
c. AB _?_ XB
d. AC _?_ XB

20. Supply reasons to complete the proof.

Given: $m\angle 2 > m\angle 1$
Prove: $m\angle 2 > m\angle 4$



Proof:

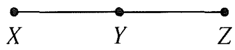
Statements	Reasons
1. $m\angle 2 > m\angle 1$	1. _?_
2. $m\angle 1 > m\angle 3$	2. _?_
3. $m\angle 2 > m\angle 3$	3. _?_
4. $\angle 3 \cong \angle 4$, or $m\angle 3 = m\angle 4$	4. _?_
5. $m\angle 2 > m\angle 4$	5. _?_

Written Exercises

Some information about the diagram is given. Tell whether the other statements can be deduced from what is given. (Write *yes* or *no*.)

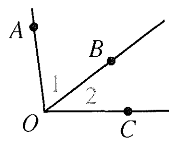
A

1. Given: Point Y lies between points X and Z.



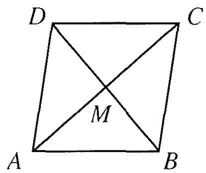
a. $XY = \frac{1}{2}XZ$ **b.** $XZ = XY + YZ$
c. $XZ > XY$ **d.** $YZ > XY$
e. $XZ > YZ$ **f.** $XZ > 2XY$

2. Given: Point B lies in the interior of $\angle AOC$.



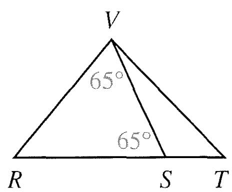
a. $m\angle 1 = m\angle 2$ **b.** $m\angle AOC = m\angle 1 + m\angle 2$
c. $m\angle AOC > m\angle 1$ **d.** $m\angle AOC > m\angle 2$
e. $m\angle 1 > m\angle 2$ **f.** $m\angle AOC > 90$

3. Given: $\square ABCD$; $AC > BD$



a. $AB > AD$ **b.** $AM > MC$
c. $DM = MB$ **d.** $AM > MB$

4. Given: $m\angle RVS = m\angle RSV = 65$

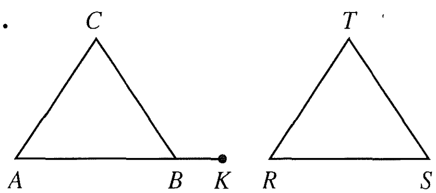


a. $RT > RS$ **b.** $RT > RV$
c. $RS > ST$ **d.** $VT < RS$

5. When some people are given that $j > k$ and $l > m$, they carelessly conclude that $j + k > l + m$. Find values for j, k, l, and m that show this conclusion is false.

Write the reasons that justify the statements.

6.

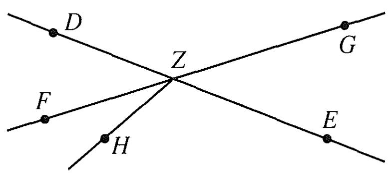


Given: $\triangle ABC \cong \triangle RST$
Prove: $AK > RS$

Statements of proof:

1. $\triangle ABC \cong \triangle RST$
2. $\overline{AB} \cong \overline{RS}$, or $AB = RS$
3. $AK = AB + BK$
4. $AK > AB$
5. $AK > RS$

7.



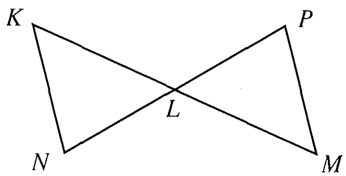
Given: $\overleftrightarrow{DE}$, $\overleftrightarrow{FG}$ and $\overrightarrow{ZH}$ contain point Z.
Prove: $m\angle DZH > m\angle GZE$

Statements of proof:

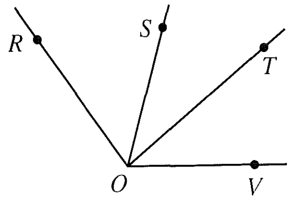
1. $\angle DZF \cong \angle GZE$, or $m\angle DZF = m\angle GZE$
2. $m\angle DZH = m\angle DZF + m\angle FZH$
3. $m\angle DZH > m\angle DZF$
4. $m\angle DZH > m\angle GZE$

Write proofs in two-column form.

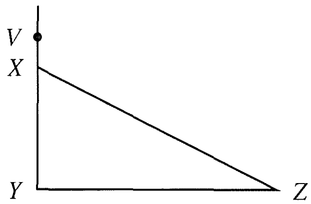
B **8.** Given: $KL > NL$; $LM > LP$
Prove: $KM > NP$



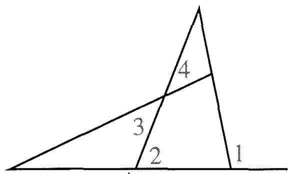
9. Given: $m\angle ROS > m\angle TOV$
Prove: $m\angle ROT > m\angle SOV$



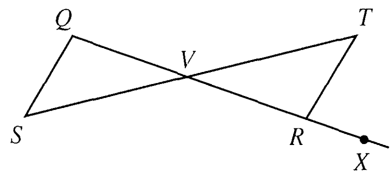
10. Given: $\overline{VY} \perp \overline{YZ}$
Prove: $\angle VXZ$ is an obtuse angle.



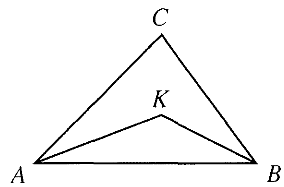
11. Given: The diagram
Prove: $m\angle 1 > m\angle 4$



12. Given: $\overline{QR}$ and $\overline{ST}$ bisect each other.
Prove: $m\angle XRT > m\angle S$

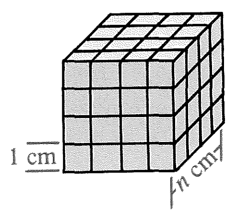


C **13.** Given: Point K lies inside $\triangle ABC$.
Prove: $m\angle K > m\angle C$



Challenge

A cube with sides n cm long is painted on all faces. It is then cut into cubes with sides 1 cm long. If $n = 4$, as the diagram at the right illustrates, how many of these smaller cubes will have paint on



a. 3 surfaces? **b.** 2 surfaces?
c. 1 surface? **d.** 0 surfaces?

Answer the questions for any positive integer n.

6-2 *Inverses and Contrapositives*

You have already studied the converse of an if-then statement. Now we consider two other related conditionals called the *inverse* and the *contrapositive*.

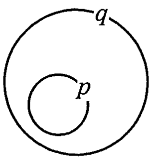
Statement:	If p, then q.
Inverse:	If not p, then not q.
Contrapositive:	If not q, then not p.

Example Write (**a**) the inverse and (**b**) the contrapositive of the true conditional: If two lines are not coplanar, then they do not intersect.

Solution **a.** Inverse: If two lines are coplanar, then they intersect. (False)
b. Contrapositive: If two lines intersect, then they are coplanar. (True)

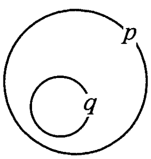
As you can see, the inverse of a true conditional is *not* necessarily true.

You can use a **Venn diagram** to represent a conditional. Since any point inside circle p is also inside circle q, this diagram represents "If p, then q." Similarly, if a point is *not* inside circle q, then it *can't* be inside circle p. Therefore, the same diagram also represents "If not q, then not p." Since the same diagram represents both a conditional and its contrapositive, these statements are either both true or both false. They are called **logically equivalent** statements.



Since a conditional and its contrapositive are logically equivalent, you may prove a conditional by proving its contrapositive. Sometimes this is easier, as you will see in Written Exercises 21 and 22.

The Venn diagram at the right represents both the converse "If q, then p" and the inverse "If not p, then not q." Therefore, the converse and the inverse of a conditional are also logically equivalent statements.

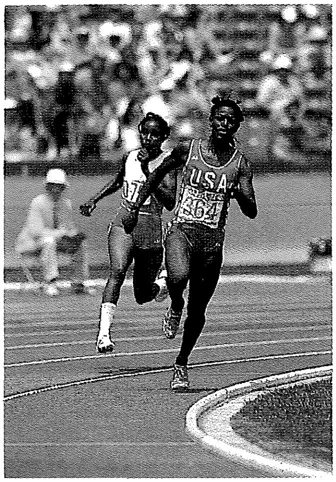


Summary of Related If-Then Statements

Given statement:	If p, then q.
Contrapositive:	If not q, then not p.
Converse:	If q, then p.
Inverse:	If not p, then not q.

A statement and its contrapositive are logically equivalent.
A statement is *not* logically equivalent to its converse or to its inverse.

Using a Venn diagram to illustrate a conditional statement can also help you determine whether an argument leads to a valid conclusion.



Suppose this conditional is true:

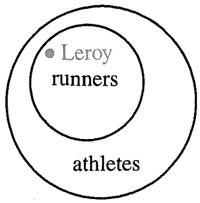
All runners are athletes.
(If a person is a runner, then that person is an athlete.)

What can you conclude from each additional statement?

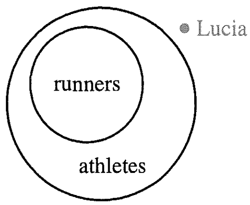
1. Leroy is a runner.
2. Lucia is not an athlete.
3. Linda is an athlete.
4. Larry is not a runner.

The conditional is paired with the four different statements as shown below.

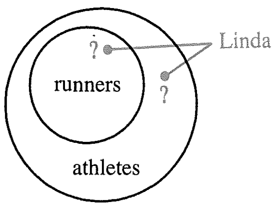
1. *Given:* If p, then q; All runners are athletes.
   p Leroy is a runner.
   Conclusion: q Leroy is an athlete.



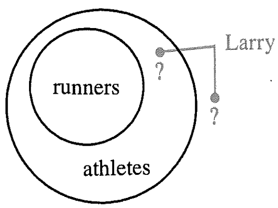
2. *Given:* If p, then q; All runners are athletes.
   not q Lucia is not an athlete.
   Conclusion: not p Lucia is not a runner.



3. *Given:* If p, then q; All runners are athletes.
   q Linda is an athlete.
   No conclusion follows. Linda might be a runner or she might not be.



4. *Given:* If p, then q; All runners are athletes.
   not p Larry is not a runner.
   No conclusion follows. Larry might be an athlete or he might not be.



Classroom Exercises

1. State the contrapositive of each statement.
   a. If I can sing, then you can dance.
   b. If you can't play baseball, then I can't ride a horse.
   c. If $x = 4$, then $x^2 - 5 = 11$.
   d. If $y < 3$, then $y \neq 4$.
   e. If a polygon is a triangle, then the sum of the measures of its angles is 180.

2. State the inverse of each statement in Exercise 1.

3. A certain conditional is true. Must its converse be true? Must its inverse be true? Must its contrapositive be true?

4. A certain conditional is false. Must its converse be false? Must its inverse be false? Must its contrapositive be false?

Classify each conditional as true or false. Then state its inverse and contrapositive, and classify each of these as true or false.

5. If a triangle is equilateral, then it is equiangular.

6. If $\angle A$ is acute, then $m\angle A \neq 100$.

7. If a triangle is not isosceles, then it is not equilateral.

8. If two planes do not intersect, then they are parallel.

Express each statement in if-then form.

9. All squares are rhombuses.

10. No trapezoids are equiangular.

11. All marathoners have stamina.

12. Suppose "All marathoners have stamina" is a true conditional. What, if anything, can you conclude from each additional statement? If no conclusion is possible, say so.
   a. Nick is a marathoner.
   b. Heidi has stamina.
   c. Mimi does not have stamina.
   d. Arlo is not a marathoner.

Written Exercises

Write (a) the contrapositive and (b) the inverse of each statement.

A **1.** If $n = 17$, then $4n = 68$.

2. If those are red and white, then this is blue.

3. If x is not even, then $x + 1$ is not odd.

4. If Abby is not here, then she is not well.

For each statement in Exercises 5–10 copy and complete a table like the one shown below.

	If ___?___, then ___?___.	True/False
Statement	?	?
Contrapositive	?	?
Converse	?	?
Inverse	?	?

5. If I live in Los Angeles, then I live in California.
6. If $\angle 1$ and $\angle 2$ are vertical angles, then $m\angle 1 = m\angle 2$.
7. If $AM = MB$, then M is the midpoint of $\overline{AB}$.
8. If a triangle is scalene, then it has no congruent sides.

B

9. If $-2n < 6$, then $n > -3$.
10. If $x^2 > 1$, then $x > 1$.

Reword the given statement in if-then form and illustrate it with a Venn diagram. What can you conclude by using the given statement together with each additional statement? If no conclusion is possible, say so.

11. Given: All senators are at least 30 years old.
    - **a.** Jose Avila is 48 years old.
    - **b.** Rebecca Castelloe is a senator.
    - **c.** Constance Brown is not a senator.
    - **d.** Ling Chen is 29 years old.
12. Given: Math teachers assign hours of homework.
    - **a.** Bridget Sullivan is a math teacher.
    - **b.** August Campos assigns hours of homework.
    - **c.** Andrew Byrnes assigns no homework at all.
    - **d.** Jason Babler is not a math teacher.

What can you conclude by using the given statement together with each additional statement? If no conclusion is possible, say so.

13. Given: If it is not raining, then I am happy.
    - **a.** I am not happy.
    - **b.** It is not raining.
    - **c.** I am overjoyed.
    - **d.** It is raining.
14. Given: All my students love geometry.
    - **a.** Stu is my student.
    - **b.** Luis loves geometry.
    - **c.** Stella is not my student.
    - **d.** George does not love geometry.
15. Given: If two angles are vertical angles, then they are congruent.
    - **a.** $\angle 1 \cong \angle 2$
    - **b.** $m\angle ABC \neq m\angle DBF$
    - **c.** $\angle 3$ and $\angle 4$ are adjacent angles.
    - **d.** $\overline{RS}$ and $\overline{TU}$ intersect at V.

What can you conclude by using the given statement together with each additional statement? If no conclusion is possible, say so.

16. Given: The diagonals of a rhombus are perpendicular.
  a. *JKLM* is a rhombus. **b.** In quad. *DIME*, $\overline{DM} \perp \overline{IE}$.
  c. *STUV* is not a rhombus. **d.** In quad. *NOPQ*, $\overline{NP} \not\perp \overline{OQ}$.

17. Given: The diagonals of a rectangle are congruent.
  a. *PQRS* is a rectangle. **b.** In quad. *ABCD*, $AC = BD$.
  c. *WXYZ* is not a rectangle. **d.** In quad. *STAR*, $SA > TR$.

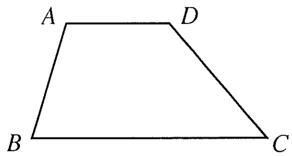
18. Given: Every square is a rhombus.
  a. *ABCD* is a rhombus. **b.** In quad. *LAST*, $LA \neq LT$.
  c. *PQRS* is a square. **d.** *GHIJ* is not a square.

C **19.** What simpler name can be used for the converse of the inverse of a conditional?

20. Write the contrapositive of the converse of the inverse of the conditional: If *r*, then *s*.

Prove each of the following statements by proving its contrapositive. Begin by writing what is given and what is to be proved.

21. If $m\angle A + m\angle B \neq 180$,
then $m\angle D + m\angle C \neq 180$.

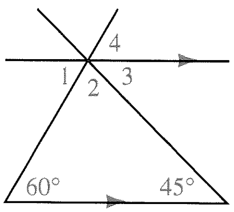


22. If n^2 is not a multiple of 3,
then *n* is not a multiple of 3.

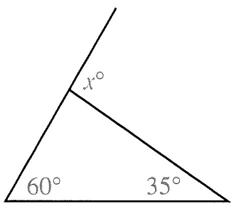
Mixed Review Exercises

Complete each statement with the word *always*, *sometimes*, or *never*.

1. Two lines that do not intersect are __?__ parallel.
2. Two lines parallel to the same plane __?__ intersect.
3. The diagonals of a parallelogram __?__ bisect each other.
4. An acute triangle is __?__ a right triangle.
5. Two lines parallel to a third line are __?__ parallel.
6. A square is __?__ a rectangle.
7. An altitude of a triangle is __?__ a median.
8. Find the measures of $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ in the figure shown.



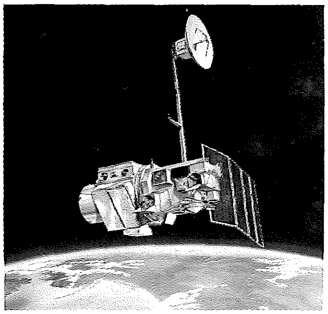
9. Find the value of *x*.

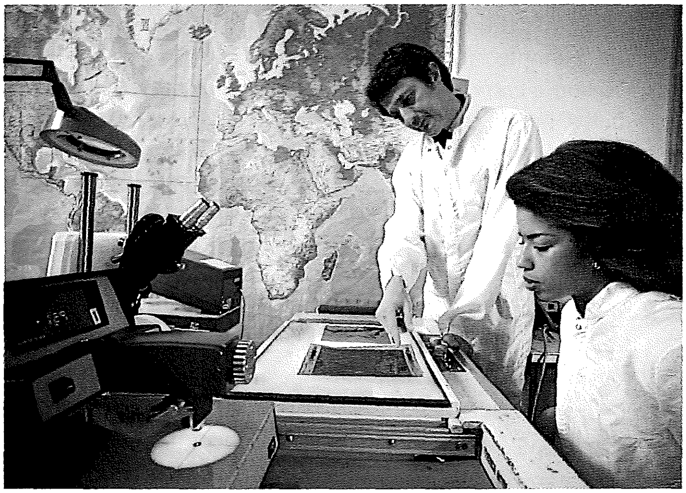


Career

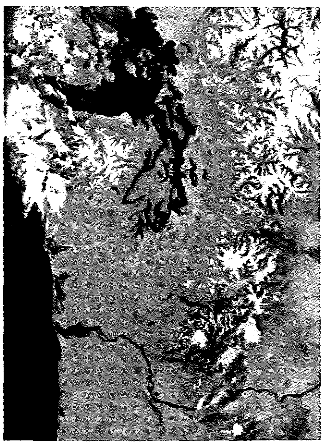
Cartographer

When you think of a map, do you think of a piece of paper with colored areas and lines? Surprisingly, some maps now consist of thousands, or even millions, of numbers stored on computer tapes. Obviously *cartography*, or map-making, is changing.





Several technical advances have led to changes in mapping. Space satellites carrying scanners produce extremely detailed images of the entire world at regular intervals. Besides conventional photographs, these scanners also record images using infrared and other wavelengths beyond the range of visible light. After processing by computer, such images provide many more kinds of information than the traditional political boundaries and topographic features of conventional maps. For example, they can map soil types and land use, distinguishing among farm fields, forests, and urban areas. In fact, they can even differentiate a corn field from a soybean field, or a freshly plowed field from a field with a mature crop.



In the false-color map shown below of Oregon and Washington, vegetation appears as red, dry regions are blue, water is black, and snow on the Cascade Mountains and the Olympic Mountains is white.

Both new images and conventional map data are being digitized, that is, converted to numerical codes and stored on computer tape. As new images are received, changes in physical features are coded and recorded. Map users can thus be provided with maps that are constantly being revised and kept up to date.

6-3 *Indirect Proof*

Until now, the proofs you have written have been *direct* proofs. Sometimes it is difficult or even impossible to find a direct proof. In that case it may be possible to reason *indirectly*. Indirect reasoning is commonplace in everyday life. Suppose, for example, that after walking home, Sue enters the house carrying a dry umbrella. You can conclude that it is not raining outside. Why? Because if it were raining, then her umbrella would be wet. The umbrella is not wet. Therefore, it is not raining.

In an **indirect proof** you begin by assuming temporarily that the desired conclusion is not true. Then you reason logically until you reach a contradiction of the hypothesis or some other known fact. Because you've reached a contradiction, you know that the temporary assumption is impossible and therefore the desired conclusion must be true.

The procedure for writing an indirect proof is summarized below. Notice how these steps are applied in the examples of indirect proof that follow.

How to Write an Indirect Proof

1. Assume temporarily that the conclusion is not true.
2. Reason logically until you reach a contradiction of a known fact.
3. Point out that the temporary assumption must be false, and that the conclusion must then be true.

Example 1

Given: n is an integer and n^2 is even.

Prove: n is even.

Proof:

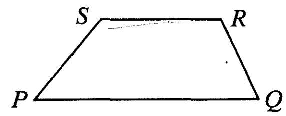
Assume temporarily that n is not even. Then n is odd, and

$$\begin{aligned} n^2 &= n \times n \\ &= \text{odd} \times \text{odd} = \text{odd}. \end{aligned}$$

But this contradicts the given information that n^2 is even. Therefore the temporary assumption that n is not even must be false. It follows that n is even.

Example 2

Prove that the bases of a trapezoid have unequal lengths.



Given: Trap. $PQRS$ with bases $\overline{PQ}$ and $\overline{SR}$

Prove: $PQ \neq SR$

Proof:

Assume temporarily that $PQ = SR$. We know that $\overline{PQ} \parallel \overline{SR}$ by the definition of a trapezoid. Since quadrilateral $PQRS$ has two sides that are both congruent and parallel, it must be a parallelogram, and $\overline{PS}$ must be parallel to $\overline{QR}$. But this contradicts the fact that, by definition, trapezoid $PQRS$ can have only one pair of parallel sides. The temporary assumption that $PQ = SR$ must be false. It follows that $PQ \neq SR$.

Can you see how proving a statement by proving its contrapositive is related to *indirect proof*? If you want to prove the statement "If p, then q," you could prove the contrapositive "If not q, then not p." Or you could write an indirect proof—assume that q is false and show that this assumption implies that p is false.

Classroom Exercises

1. An indirect proof is to be used to prove the following:

$$\text{If } AB = AC, \text{ then } \triangle ABD \cong \triangle ACD.$$

Which one of the following is the correct way to begin?

a. Assume temporarily that $AB \neq AC$.

b. Assume temporarily that $\triangle ABD \not\cong \triangle ACD$.

What is the first sentence of an indirect proof of the statement shown?

2. $\triangle ABC$ is equilateral.

3. Doug is a Canadian.

4. $a \geq b$

5. Kim isn't a violinist.

6. $m\angle X > m\angle Y$

7. $\overline{CX}$ isn't a median of $\triangle ABC$.

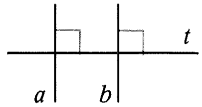
8. Planning to write an indirect proof that $\angle A$ is an obtuse angle, Becky began by saying "Assume temporarily that $\angle A$ is an acute angle." What has Becky overlooked?

9. Wishing to prove that l and m are skew lines, John began an indirect proof by supposing that l and m are intersecting lines. What possibility has John overlooked?

10. Arrange sentences (a)–(e) in an order that completes an indirect proof of the following statement: In a plane, two lines perpendicular to a third line are parallel to each other.

Given: Lines a, b, and t lie in a plane;
$a \perp t$; $b \perp t$

Prove: $a \parallel b$



(a) Then a intersects b in some point Z.
(b) But this contradicts the theorem which says that there is exactly one line perpendicular to a given line through a point outside the line.
(c) It is false that a is not parallel to b, and it follows that $a \parallel b$.
(d) Assume temporarily that a is not parallel to b.
(e) Then there are two lines through Z and perpendicular to t.

Written Exercises

Suppose someone plans to write an indirect proof of each conditional. Write a correct first sentence of the indirect proof.

A **1.** If $m\angle A = 50$, then $m\angle B = 40$.

2. If $\overline{DF} \not\cong \overline{RT}$, then $\overline{DE} \not\cong \overline{RS}$.

3. If $a \neq b$, then $a - b \neq 0$.

4. If $x^2 \neq y^2$, then $x \neq y$.

5. If $\overline{EF} \not\cong \overline{GH}$, then $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ aren't parallel.

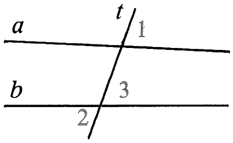
Write an indirect proof in paragraph form.

6. Given: People wearing coats are shivering as they come to the door.
Prove: It's cold outside.

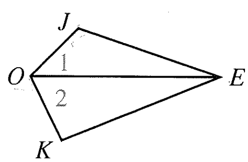
7. Given: $\triangle XYZ$; $m\angle X = 100$
Prove: $\angle Y$ is not a rt. $\angle$.

8. Given: n is an integer and n^2 is odd.
Prove: n is odd.

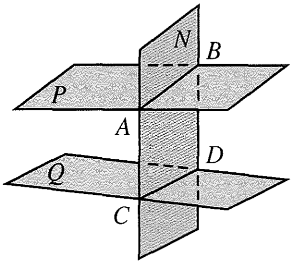
9. Given: Transversal t cuts lines a and b;
$m\angle 1 \neq m\angle 2$
Prove: $a \nparallel b$



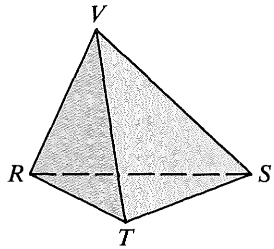
10. Given: $\overline{OJ} \cong \overline{OK}$; $\overline{JE} \not\cong \overline{KE}$
Prove: $\overrightarrow{OE}$ doesn't bisect $\angle JOK$.



B **11.** Given: $\overleftrightarrow{AB} \nparallel \overleftrightarrow{CD}$
Prove: Planes P and Q intersect.

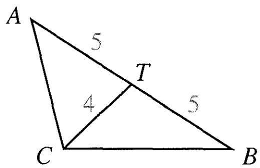


12. Given: $\triangle RVT$ and SVT are equilateral; $\triangle RVS$ is not equilateral.
Prove: $\triangle RST$ is not equilateral.

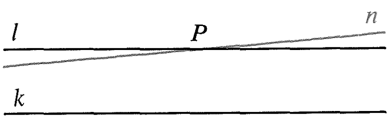


13. Given: quad. $EFGH$ in which $m\angle EFG = 93$; $m\angle FGH = 20$; $m\angle GHE = 147$; $m\angle HEF = 34$
Prove: $EFGH$ is not a convex quadrilateral.

14. Given: $AT = BT = 5$; $CT = 4$
Prove: $\angle ACB$ is not a rt. $\angle$.



15. Given: Coplanar lines l, k, n; n intersects l in P; $l \parallel k$
Prove: n intersects k.



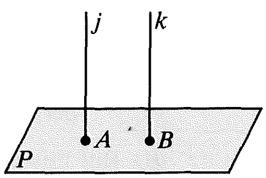
16. Prove that if two angles of a triangle are not congruent, then the sides opposite those angles are not congruent.

17. Prove that there is no regular polygon with an interior angle whose measure is 155.

18. Prove that the diagonals of a trapezoid do not bisect each other.

C **19.** Prove that if two lines are perpendicular to the same plane, then the lines do not intersect.

20. Given: Points R, S, T, and W; $\overleftrightarrow{RS}$ and $\overleftrightarrow{TW}$ are skew.
Prove: $\overleftrightarrow{RT}$ and $\overleftrightarrow{SW}$ are skew.



Ex. 19

Challenge

One of four children ate the last piece of lasagna. When questioned they responded as follows:

Joan: I didn't eat it.
Ken: Leo ate it.
Leo: Martha ate it.
Martha: Leo is lying.

If only one of the four children lied, who ate the last piece?

Self-Test 1

Classify each conditional as true or false.

1. If $j > k$, then $k < j$.

2. If $a > b$ and $b = c$, then $a > c$.

3. If $r > t$ and $s > t$, then $r > s$.

4. If $\angle BCD$ is an exterior angle of $\triangle ABC$, then $m\angle BCD > m\angle A + m\angle B$.

Use the conditional: If $\triangle ABC$ is acute, then $m\angle C \neq 90$.

5. Write the inverse of the statement. Is it true or false?

6. Write the contrapositive of the statement. Is it true or false?

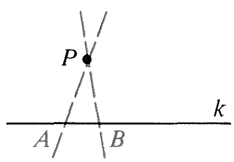
7. Write the letter paired with the statement that is logically equivalent to "If Dan can't go, then Valerie can go."
A. If Valerie can go, then Dan can't go.
B. If Dan can go, then Valerie can't go.
C. If Valerie can't go, then Dan can go.
D. If Dan can go, then Valerie can go.

8. Given: All rhombuses are parallelograms.
What can you conclude from each additional statement? If no conclusion is possible, write *no conclusion*.
a. *ABCD* is not a parallelogram.
b. *QRST* is not a rhombus.
c. *MNOP* is a parallelogram.
d. *GHIJ* is a rhombus.

9. Suppose you plan to write an indirect proof of the statement: If $AB = 7$, then $AC = 14$. Write a correct first sentence of the indirect proof.

10. Write the letters (a)–(d) in an order that completes an indirect proof of the statement: Through a point outside a line, there is at most one line perpendicular to the given line.

Given: Point P not on line k
Prove: There is at most one line through P perpendicular to k.



(a) But this contradicts Corollary 3 of Theorem 3-11: In a triangle, there can be at most one right angle or obtuse angle.
(b) Then $\angle PAB$ and $\angle PBA$ are right angles, and $\triangle PAB$ has two right angles.
(c) Thus our temporary assumption is false, and there is at most one line through P perpendicular to k.
(d) Assume temporarily that there are two lines through P perpendicular to k at A and B.

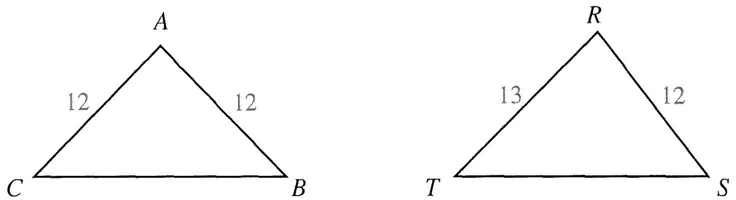
Inequalities in Triangles

Objectives

1. State and apply the inequality theorems and corollaries for one triangle.
2. State and apply the inequality theorems for two triangles.

6-4 *Inequalities for One Triangle*

From the information given in the diagram at the left below you can deduce that $\angle C \cong \angle B$.



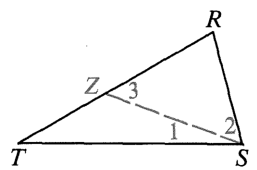
Using the information in the diagram at the right you could write an indirect proof showing that $m\angle S \neq m\angle T$. The following theorem enables you to reach an even stronger conclusion, the conclusion that $m\angle S > m\angle T$.

Theorem 6-2

If one side of a triangle is longer than a second side, then the angle opposite the first side is larger than the angle opposite the second side.

Given: $\triangle RST$; $RT > RS$

Prove: $m\angle RST > m\angle T$

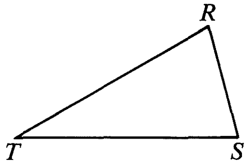


Proof:

By the Ruler Postulate there is a point Z on $\overline{RT}$ such that $RZ = RS$. Draw $\overline{SZ}$.
In isosceles $\triangle RZS$, $m\angle 3 = m\angle 2$.
Because $m\angle RST = m\angle 1 + m\angle 2$, you have $m\angle RST > m\angle 2$.
Substitution of $m\angle 3$ for $m\angle 2$ yields $m\angle RST > m\angle 3$.
Because $\angle 3$ is an ext. $\angle$ of $\triangle ZST$, you have $m\angle 3 > m\angle T$.
From $m\angle RST > m\angle 3$ and $m\angle 3 > m\angle T$, you get $m\angle RST > m\angle T$.

Theorem 6-3

If one angle of a triangle is larger than a second angle, then the side opposite the first angle is longer than the side opposite the second angle.



Given: $\triangle RST$; $m\angle S > m\angle T$

Prove: $RT > RS$

Proof:

Assume temporarily that $RT \not> RS$. Then either $RT = RS$ or $RT < RS$.

Case 1: If $RT = RS$, then $m\angle S = m\angle T$.

Case 2: If $RT < RS$, then $m\angle S < m\angle T$ by Theorem 6-2.

In either case there is a contradiction of the given fact that $m\angle S > m\angle T$. The assumption that $RT \not> RS$ must be false. It follows that $RT > RS$.

Corollary 1

The perpendicular segment from a point to a line is the shortest segment from the point to the line.

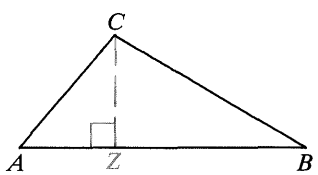
Corollary 2

The perpendicular segment from a point to a plane is the shortest segment from the point to the plane.

See Classroom Exercises 18 and 19 for proofs of the corollaries.

Theorem 6-4 *The Triangle Inequality*

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



Given: $\triangle ABC$

Prove: (1) $AB + BC > AC$
(2) $AB + AC > BC$
(3) $AC + BC > AB$

Proof:

One of the sides, say $\overline{AB}$, is the longest side. (Or $\overline{AB}$ is at least as long as each of the other sides.) Then (1) and (2) are true. To prove (3), draw a line, $\overleftrightarrow{CZ}$, through C and perpendicular to $\overleftrightarrow{AB}$. (Through a point outside a line, there is exactly one line perpendicular to the given line.) By Corollary 1 of Theorem 6-3, $\overline{AZ}$ is the shortest segment from A to $\overline{CZ}$. Also, $\overline{BZ}$ is the shortest segment from B to $\overline{CZ}$. Therefore

$AC > AZ$ and $BC > ZB$.
$AC + BC > AZ + ZB$ (Why?)
$AC + BC > AB$ (Why?)

Example The lengths of two sides of a triangle are 3 and 5. The length of the third side must be greater than _?_, but less than _?_.

Solution Let x be the length of the third side.

$$x + 3 > 5 \qquad 3 + 5 > x \qquad x + 5 > 3$$
$$x > 2 \qquad 8 > x \qquad x > -2$$

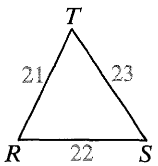
The length of the third side must be greater than 2 but less than 8.

Note that the inequality $x + 5 > 3$ did not give us any useful information. Since $5 > 3$, the sum of 5 and *any* positive number is greater than 3.

Classroom Exercises

Name the largest angle and the smallest angle of the triangle.

1.



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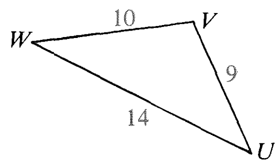
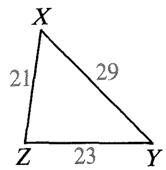
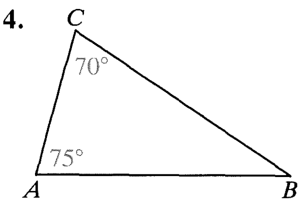


3.

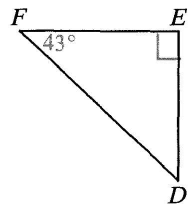


Name the longest side and the shortest side of the triangle.

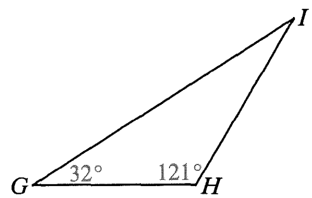
4.



5.



6.



Is it possible for a triangle to have sides with the lengths indicated?

7. 10, 9, 8
8. 6, 6, 20
9. 7, 7, 14.1
10. 16, 11, 5
11. 0.6, 0.5, 1
12. 18, 18, 0.06

13. An isosceles triangle is to have a base that is 20 cm long. Draw a diagram to show the following.
  a. The legs can be very long.
  b. Although the legs must be more than 10 cm long, each length can be very close to 10 cm.

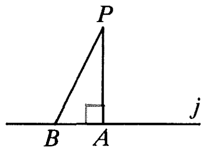
14. The base of an isosceles triangle has length 12. What can you say about the length of a leg?

15. Two sides of a parallelogram have lengths 10 and 12. What can you say about the lengths of the diagonals?

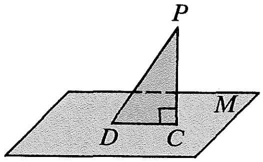
16. Two sides of a triangle have lengths 15 and 20. The length of the third side can be any number between _?_ and _?_.

17. Suppose you know only that the length of one side of a rectangle is 100. What can you say about the length of a diagonal?

18. Use the diagram below to explain how Corollary 1 follows from Theorem 6-3.



Ex. 18



Ex. 19

19. Use the diagram, in which $\overline{PC} \perp$ plane M, to explain how Corollary 2 follows from Theorem 6-3 or from Corollary 1.

20. Which is the largest angle of a right triangle? Which is the longest side of a right triangle? Explain.

Written Exercises

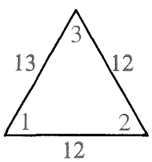
The lengths of two sides of a triangle are given. Write the numbers that best complete the statement: The length of the third side must be greater than _?_, but less than _?_.

A **1.** 6, 9 **2.** 15, 13 **3.** 100, 100

4. $7n$, $10n$ **5.** a, b (where $a > b$) **6.** k, $k + 5$

In Exercises 7–9 the diagrams are not drawn to scale. If each diagram were drawn to scale, which numbered angle would be the largest?

7.



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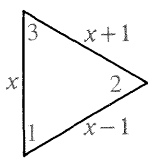

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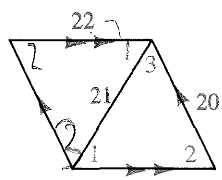
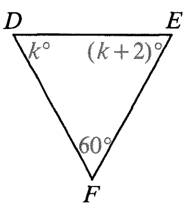
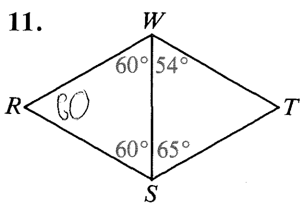


In Exercises 10–14 the diagrams are not drawn to scale. If each diagram were drawn to scale, which segment shown would be the longest?

10.



11.



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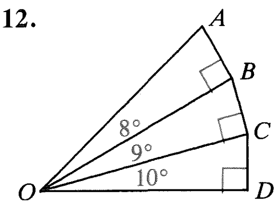
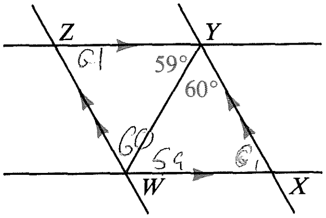
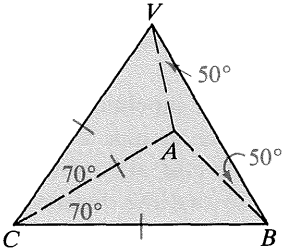


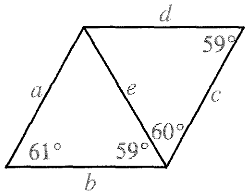
B **13.**



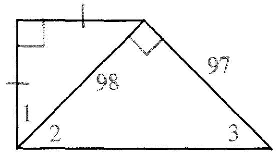
14.



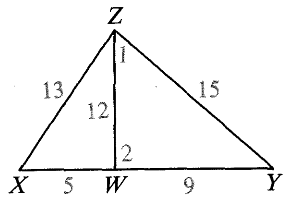
15. Use the lengths a, b, c, d, and e to complete:
$\underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ }$



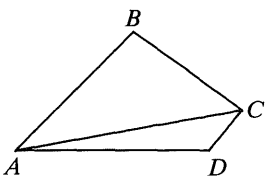
16. Use $m\angle 1$, $m\angle 2$, and $m\angle 3$ to complete:
$\underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ }$



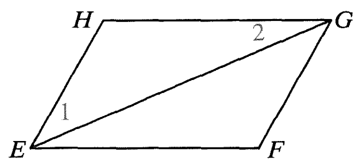
17. The diagram is not drawn to scale. Use $m\angle 1$, $m\angle 2$, $m\angle X$, $m\angle Y$, and $m\angle XZY$ to complete:
$\underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ } > \underline{\ ?\ }$



18. Given: Quad. $ABCD$
Prove: $AB + BC + CD + DA > 2(AC)$



19. Given: $\square EFGH$; $EF > FG$
Prove: $m\angle 1 > m\angle 2$



C **20.** Discover, state, and prove a theorem that compares the perimeter of a quadrilateral with the sum of the lengths of the diagonals.

21. Prove that the sum of the lengths of the medians of a triangle is greater than half the perimeter.

22. If you replace "medians" with "altitudes" in Exercise 21, can you prove the resulting statement? Explain.

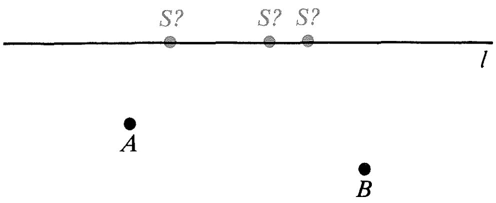
In Exercises 23 and 24, begin your proofs by drawing auxiliary lines.

23. Discover, state, and prove a theorem that compares the length of the longest side of a quadrilateral with the sum of the lengths of the other three sides.

24. Prove: If P is any point inside $\triangle XYZ$, then $ZX + ZY > PX + PY$.

Application — *Finding the Shortest Path*

The owners of pipeline l plan to construct a pumping station at a point S on line l in order to pipe oil to two major customers, located at A and B. To minimize the cost of constructing lines from S to A and B, they wish to locate S along l so that the distance $SA + SB$ is as small as possible.



The construction engineer uses the following method to locate S:

1. Draw a line through B perpendicular to l, intersecting l at point P.
2. On this perpendicular, locate point C so that $PC = PB$.
3. Draw $\overline{AC}$.
4. Locate S at the intersection of $\overline{AC}$ and l.

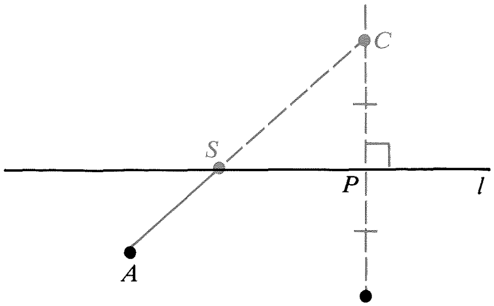


Figure 1

Figure 2 shows the path of the new pipelines through the pumping station located at S, and an alternative path going through a different point, X, on l. You can use Theorem 6-4 (the Triangle Inequality) to show that if X is any point on l other than S, then $AX + XB > AS + SB$. So any alternative path is longer than the path through S.

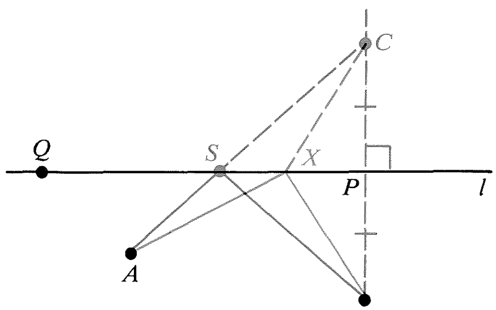


Figure 2

Exercises

Exercises 1 and 2 refer to Figure 2 on the preceding page.

1. Supply the reason for each key step of the proof that the method given for finding S yields the shortest total length for the pipelines serving A and B.

1. l is the perpendicular bisector of $\overline{BC}$.
2. $SC = SB$
3. $AS + SC = AC$
4. $AS + SB = AC$
5. $XC = XB$
6. $AX + XC > AC$
7. $AX + XB > AS + SB$

2. This method for finding S is sometimes called a *solution by reflection*, since it involves *reflecting* point B in line l. (See Chapter 14 for more on reflections.) Show that $\overline{AS}$ and $\overline{SB}$, like reflected paths of light, make congruent angles with l. That is, prove that $\angle QSA \cong \angle PSB$. (*Hint*: Draw your own diagram, omitting the part of Figure 2 shown in blue.)

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

Draw several *pairs* of triangles, varying the size of just one side or just one angle. Make charts like the ones below to record your data. Record the lengths of the sides and measures of the angles you give, as well as the measurements you get. Do as many pairs as you need to help you recognize a pattern.

Enter the lengths of all three sides (SSS).

pair 1	
$AB =$ $BC =$ $AC =$	same $AB =$ same $BC =$ longer $AC =$
$m\angle ABC =$	$m\angle ABC =$

What happened to the angle opposite the side you made longer?

Enter the lengths of two sides and the included angle (SAS).

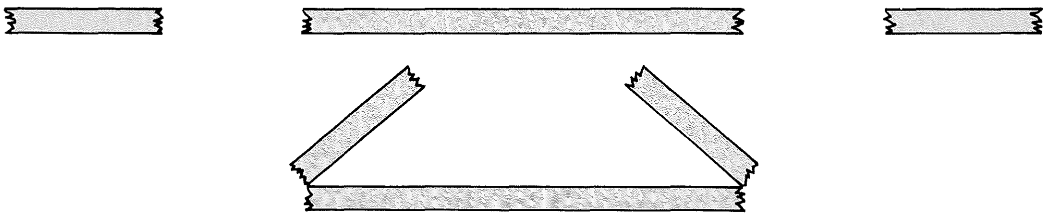
pair 1	
$AB =$ $\angle BAC =$ $AC =$	same $AB =$ larger $\angle BAC =$ same $AC =$
$BC =$	$BC =$

What happened to the side opposite the angle you made larger?

♦ Computer Key-In

If you break a stick into three pieces, what is the probability that you can join the pieces end-to-end to form a triangle?

It's easy to see that if the sum of the lengths of any two of the pieces is less than or equal to that of the third, a triangle can't be formed. This is the Triangle Inequality (Theorem 6-4).

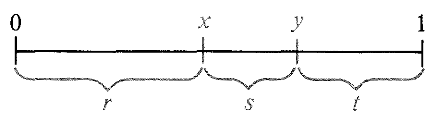


By an experiment, your class can estimate the probability that three pieces of broken stick will form a triangle. Suppose everyone in your class has a stick 1 unit long and breaks it into three pieces. If there are thirty people in your class and eight people are able to form a triangle with their pieces, we estimate that the probability of forming a triangle is about $\frac{8}{30}$, or $\frac{4}{15}$.

Of course, this experiment is not very practical. You can get much better results by having a computer simulate the breaking of many, many sticks, as in the program in BASIC on the next page.

Computer simulations are useful whenever large numbers of operations need to be done in a short period of time. In this problem, for example, an accurate probability depends on using a large number of sticks. Computer simulations have been used when a real experiment would be costly or dangerous; aeronautics companies use real-time flight simulators on the ground to train pilots. Simulations are also applied to investigate statistical data where many variables determine the outcome, as in the analysis and prediction of weather patterns. In the stick-triangle problem, using the computer has another advantage—a computer can break very small pieces that a human couldn't, so the probability figure will be theoretically more accurate, if less "realistic."

In lines 30 and 40 of the following program, you tell the computer how many sticks you want to break. Each stick is 1 unit long, and the computer breaks each stick by choosing two random numbers x and y between 0 and 1. These numbers divide the stick into three lengths r, s, and t.



The computer then keeps count of the number of sticks (N) which form a triangle when broken.

Notice that RND is used in lines 70 and 80. Since usage of RND varies, check this with the manual for your computer and make any necessary changes. The computer print-outs shown in this text use capital letters. The x, y, r, s, and t used in the discussion above appear as X, Y, R, S, T.

```
10 PRINT "SIMULATION--BREAKING STICKS TO MAKE TRIANGLES"
20 PRINT
30 PRINT "HOW MANY STICKS DO YOU WANT TO BREAK";
40 INPUT D
50 LET N = 0
60 FOR I = 1 TO D
70 LET X = RND (1)
80 LET Y = RND (1)
90 IF X > = Y THEN 70
100 LET R = X
110 LET S = Y - R
120 LET T = 1 - R - S
130 IF R + S < = T THEN 170
140 IF S + T < = R THEN 170
150 IF R + T < = S THEN 170
160 LET N = N + 1
170 NEXT I
180 LET P = N/D
190 PRINT
200 PRINT "THE EXPERIMENTAL PROBABILITY THAT"
210 PRINT "A BROKEN STICK CAN FORM A TRIANGLE IS ";P
220 END
```

Line Number	Explanation
60–120	These lines simulate the breaking of each stick. When I = 10, for example, the computer is "breaking" the tenth stick.
130–150	Here the computer uses the Triangle Inequality to check whether the pieces of the broken stick can form a triangle. If not, the computer goes on to the next stick (line 170) and the value of N is not affected.
160	If the broken stick has survived the tests of steps 130–150, then the pieces can form a triangle and the value of N is increased by 1.
170	Lines 60–170 form a loop that is repeated D times. After I = D, the probability P is calculated and printed (lines 180–210).

Exercises

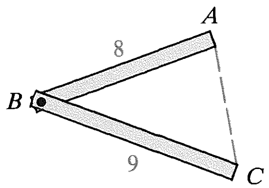
1. Pick any two numbers x and y between 0 and 1 with $x < y$. With paper and pencil, carry out the instructions in lines 100 through 150 of the program to see how the computer finds r, s, and t and tests to see whether the values can be the lengths of the sides of a triangle.

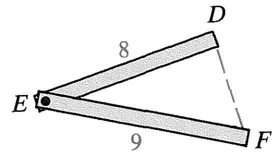
2. If you use a language other than BASIC, write a similar program for your computer.

3. Run the program several times for large values of D, say 100, 400, 800, and compare your results with those of some classmates. Does the probability that the pieces of a broken stick form a triangle appear to be less than or greater than $\frac{1}{2}$?

6-5 *Inequalities for Two Triangles*

Begin with two matched pairs of sticks joined loosely at B and E. Open them so that $m\angle B > m\angle E$ and you find that $AC > DF$. Conversely, if you open them so that $AC > DF$, you see that $m\angle B > m\angle E$. Two theorems are suggested by these examples. The first theorem is surprisingly difficult to prove. The second theorem has an indirect proof.



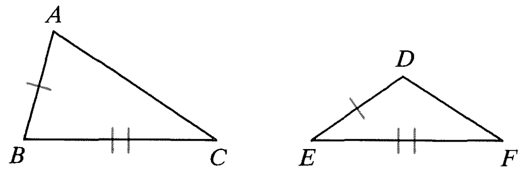


Theorem 6-5 *SAS Inequality Theorem*

If two sides of one triangle are congruent to two sides of another triangle, but the included angle of the first triangle is larger than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle.

Given: $\overline{BA} \cong \overline{ED}$; $\overline{BC} \cong \overline{EF}$;
$m\angle B > m\angle E$

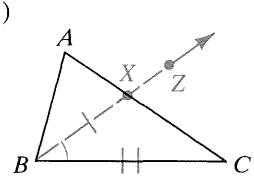
Prove: $AC > DF$



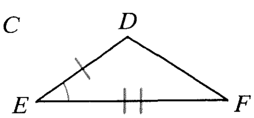
Proof:

Draw $\overrightarrow{BZ}$ so that $m\angle ZBC = m\angle E$. On $\overrightarrow{BZ}$ take point X so that $BX = ED$.
Then either X is on $\overline{AC}$ or X is not on $\overline{AC}$.
In both cases $\triangle XBC \cong \triangle DEF$ by SAS, and $XC = DF$.

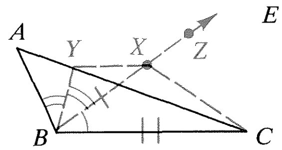
Case 1: X is on $\overline{AC}$.
$AC > XC$ (Seg. Add. Post. and a Prop. of Ineq.)
$AC > DF$ (Substitution Property, using the equation in red above)



Case 1



Case 2: X is not on $\overline{AC}$.
Draw the bisector of $\angle ABX$, intersecting $\overline{AC}$ at Y.
Draw $\overline{XY}$ and $\overline{XC}$.
$BA = ED = BX$
Since $\triangle ABY \cong \triangle XBY$ (SAS), $AY = XY$.
$XY + YC > XC$ (Why?)
$AY + YC > XC$ (Why?), or $AC > XC$
$AC > DF$ (Substitution Property)



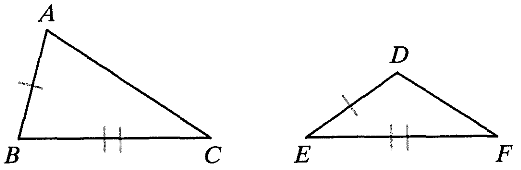
Case 2

Theorem 6-6 *SSS Inequality Theorem*

If two sides of one triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

Given: $\overline{BA} \cong \overline{ED}$; $\overline{BC} \cong \overline{EF}$; $AC > DF$

Prove: $m\angle B > m\angle E$



Proof:

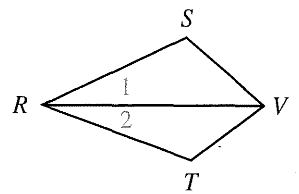
Assume temporarily that $m\angle B \not> m\angle E$.
Then either $m\angle B = m\angle E$ or $m\angle B < m\angle E$.

Case 1: If $m\angle B = m\angle E$, then $\triangle ABC \cong \triangle DEF$ by the SAS Postulate, and $AC = DF$.

Case 2: If $m\angle B < m\angle E$, then $AC < DF$ by the SAS Inequality Theorem.

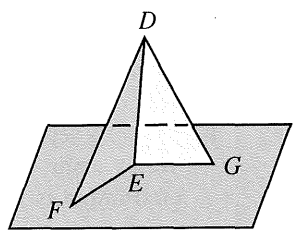
In both cases there is a contradiction of the given fact that $AC > DF$. What was temporarily assumed to be true, that $m\angle B \not> m\angle E$, must be false. It follows that $m\angle B > m\angle E$.

Example 1 Given: $\overline{RS} \cong \overline{RT}$; $m\angle 1 > m\angle 2$
What can you deduce?



Solution In the two triangles you have $\overline{RV} \cong \overline{RV}$ as well as $\overline{RS} \cong \overline{RT}$. Since $m\angle 1 > m\angle 2$, you can apply the SAS Inequality Theorem to get $SV > TV$.

Example 2 Given: $\overline{EF} \cong \overline{EG}$; $DF > DG$
What can you deduce?

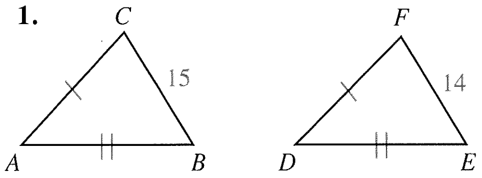


Solution $\overline{DE}$ and $\overline{EF}$ of $\triangle DEF$ are congruent to $\overline{DE}$ and $\overline{EG}$ of $\triangle DEG$. Since $DF > DG$, you can apply the SSS Inequality Theorem to get $m\angle DEF > m\angle DEG$.

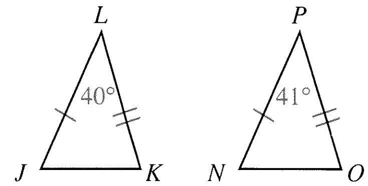
Classroom Exercises

What can you deduce? Name the theorem that supports your answer.

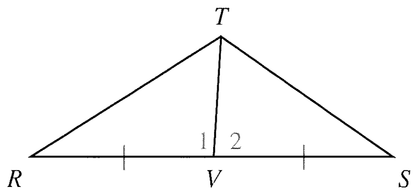
1.



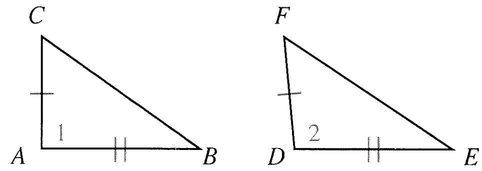
2.



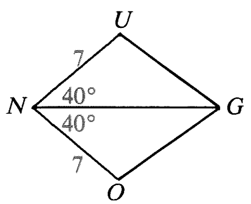
3. $m\angle 1 > m\angle 2$



4. $\angle 1$ is a rt. $\angle$; $\angle 2$ is an obtuse $\angle$.



5.



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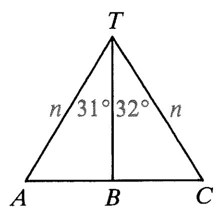

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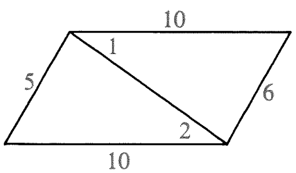

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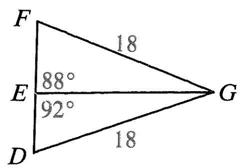


You will need a centimeter ruler and a protractor for Exercises 9 and 10.

9. a. Draw an isosceles triangle with legs 7 cm long and a vertex angle of 120°. Measure the length of the base of the triangle.

b. If you keep the legs at 7 cm in length but halve the measure of the vertex angle to 60°, what happens to the length of the base? What kind of triangle is this new triangle? What is the length of the third side?

10. a. Draw a right triangle with legs of 6 cm and 8 cm. Measure the length of the hypotenuse.

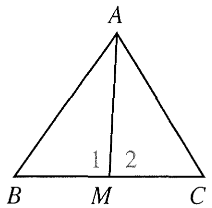
b. If you keep the 6 cm and 8 cm sides the same lengths but halve the measure of the included angle to 45°, what happens to the length of the third side? Test your answer by drawing the new triangle.

Written Exercises

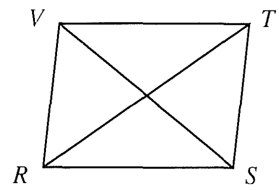
What can you deduce? Name the theorem that supports your answer.

A

1. Given: $\overline{AM}$ is a median of $\triangle ABC$;
$AB > AC$

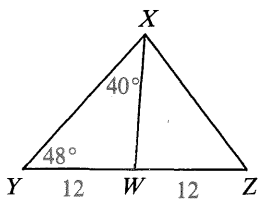


2. Given: ▱$RSTV$;
$m\angle TSR > m\angle VRS$

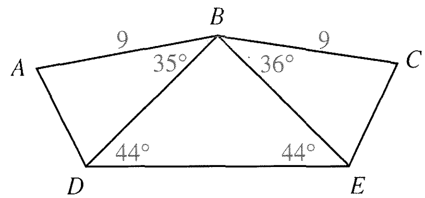


Complete the statements by writing <, =, or >.

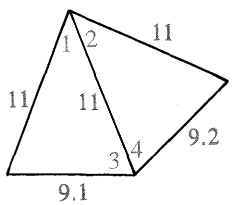
3. XY __?__ XZ;
XW __?__ 12



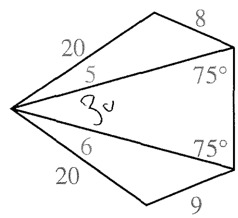
4. AD __?__ CE



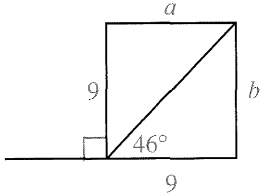
5. $m\angle 1$ __?__ $m\angle 2$;
$m\angle 3$ __?__ $m\angle 4$



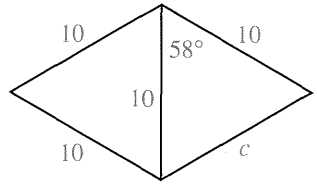
6. $m\angle 5$ __?__ $m\angle 6$



7. a __?__ b

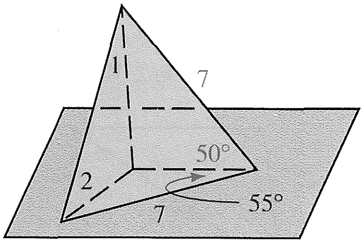


8. c __?__ 10

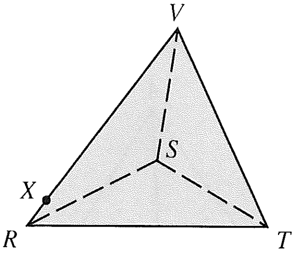


Complete the statements by writing <, =, or >.

B **9.** $m\angle 1$ __?__ $m\angle 2$

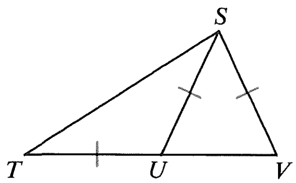


10. $SR = ST$; $VX = VT$
$m\angle RSV$ __?__ $m\angle TSV$

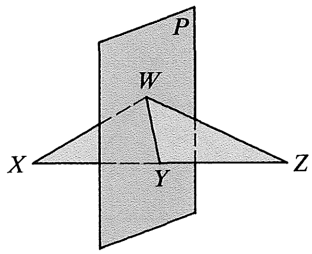


Write proofs in two-column form.

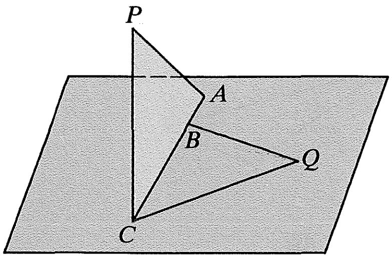
11. Given: $\overline{TU} \cong \overline{US} \cong \overline{SV}$
Prove: $ST > SV$



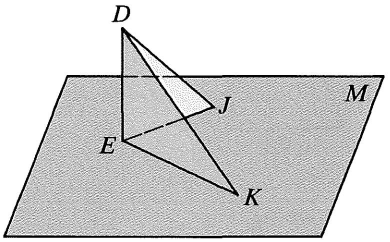
12. Given: Plane P bisects $\overline{XZ}$ at Y; $WZ > WX$
Discover and prove something about the figure.



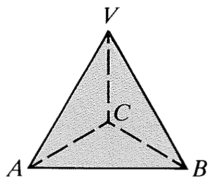
C **13.** Given: $\overline{PA} \cong \overline{PC} \cong \overline{QC} \cong \overline{QB}$
Prove: $m\angle PCA < m\angle QCB$



14. Given: $\overline{DE} \perp$ plane M; $EK > EJ$
Prove: $DK > DJ$
(*Hint*: On $\overline{EK}$, take Z so that $EZ = EJ$.)



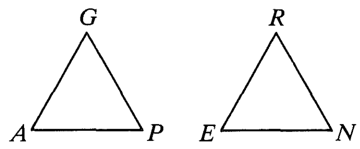
15. In the three-dimensional figure shown, all the edges *except* $\overline{VC}$ are congruent. What can you say about the measures of the largest angles of the twelve angles in the figure
a. if $\overline{VC}$ is longer than the other edges?
b. if $\overline{VC}$ is shorter than the other edges?



Self-Test 2

1. In $\triangle XYZ$, $m\angle X = 50$, $m\angle Y = 60$, and $m\angle Z = 70$. Name the longest side of the triangle.
2. In $\triangle DOM$, $\angle O$ is a right angle and $m\angle D > m\angle M$. Which side of $\triangle DOM$ is the shortest side?

Complete each statement by writing <, =, or >.



Exs. 3-5

3. If $ER > EN$, then $m\angle R$ _?_ $m\angle N$.
4. If $\overline{AG} \cong \overline{ER}$, $\overline{AP} \cong \overline{EN}$, and $\angle A \cong \angle E$, then GP _?_ RN.
5. If $\overline{GA} \cong \overline{RE}$, $\overline{GP} \cong \overline{RN}$, and $AP > EN$, then $m\angle G$ _?_ $m\angle R$.
6. The lengths of the sides of a triangle are 5, 6, and x. Then x must be greater than _?_ and less than _?_.

The longer diagonal of $\square QRST$ is $\overline{QS}$. Tell whether each statement *must be*, *may be*, or *cannot be* true.

7. $\angle R$ is an acute angle.
8. $QS > RS$
9. $RS > RT$

Extra — *Non-Euclidean Geometries*

When you develop a geometry, you have some choice as to which statements you are going to postulate and which you are going to prove. For example, consider these two statements:

(A) If two parallel lines are cut by a transversal, then corresponding angles are congruent.

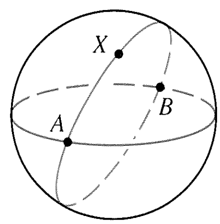
(B) Through a point outside a line, there is exactly one line parallel to the given line.

In this book, statement (A) is Postulate 10 and statement (B) is Theorem 3-8. In some books, statement (B) is a postulate (commonly called Euclid's *Parallel Postulate*) and statement (A) is a theorem. In still other developments, both of these statements are proved on the basis of some third statement chosen as a postulate.

A geometry that provides for a unique parallel to a line through a point not on the line is called *Euclidean*, so this text is a Euclidean geometry book. In the nineteenth century, it was discovered that geometries exist in which the Parallel Postulate is *not true*. Such geometries are called *non-Euclidean*. The statements at the top of the next page show the key differences between Euclidean geometry and two types of non-Euclidean geometry.

Euclidean geometry	Through a point outside a line, there is *exactly one* line parallel to the given line.
Hyperbolic geometry	Through a point outside a line, there is *more than one* line parallel to the given line. (This geometry was discovered by Bolyai, Lobachevsky, and Gauss.)
Elliptic geometry	Through a point outside a line, there is *no* line parallel to the given line. (This geometry was discovered by Riemann and is used by ship and airplane navigators.)

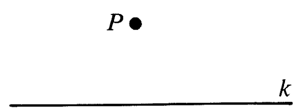
To see a model of a no-parallel geometry, visualize the surface of a sphere. Think of a line as being a great circle of the sphere, that is, the intersection of the sphere and a plane that passes through the center of the sphere. On the sphere, through a point outside a line, there is no line parallel to the given line. All lines, as defined, intersect. In the figure, for example, X is a point not on the red great circle. A line has been drawn through X, namely the great circle shown in blue. You can see that the two lines intersect in *two* points, A and B.



To see how statement (B) follows from our postulates, notice that Postulates 10 and 11 play a crucial role in the following proof. In fact, without such assumptions about parallels there couldn't be a proof. Before the discovery of non-Euclidean geometries people didn't know that this was the case and tried, without success, to find a proof that was independent of any assumption about parallels.

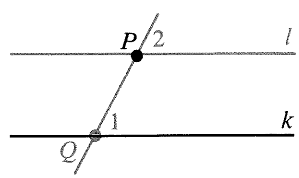
Given: Point P outside line k.

Prove: (1) There is a line through P parallel to k.
(2) There is only one line through P parallel to k.



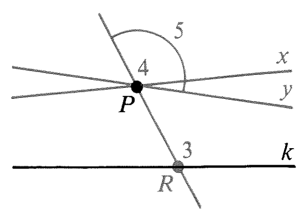
Key steps of proof of (1):

1. Draw a line through P and some point Q on k. (Postulates 5 and 6)
2. Draw line l so that $\angle 2$ and $\angle 1$ are corresponding angles and $m\angle 2 = m\angle 1$. (Protractor Postulate)
3. $l \parallel k$, so there is a line through P parallel to k. (Postulate 11)



Indirect proof of (2):

Assume temporarily that there are at least two lines, x and y, through P parallel to k. Draw a line through P and some point R on k. $\angle 4 \cong \angle 3$ and $\angle 5 \cong \angle 3$ by Postulate 10, so $\angle 5 \cong \angle 4$. But since x and y are different lines we also have $m\angle 5 > m\angle 4$. This is impossible, so our assumption must be false, and it follows that there is only one line through P parallel to k.



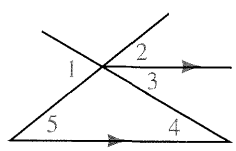
Chapter Summary

1. The properties of inequality most often used are stated on page 204.
2. The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle. (The Exterior Angle Inequality Theorem)
3. The summary on page 208 gives the relationship between an if-then statement, its converse, its inverse, and its contrapositive. An if-then statement and its contrapositive are logically equivalent.
4. You begin an indirect proof by assuming temporarily that what you wish to prove true is *not* true. If this temporary assumption leads to a contradiction of a known fact, then your temporary assumption must be false and what you wish to prove true must be true.
5. In $\triangle RST$, if $RT > RS$, then $m\angle S > m\angle T$. Conversely, if $m\angle S > m\angle T$, then $RT > RS$.
6. The perpendicular segment from a point to a line (or plane) is the shortest segment from the point to the line (or plane).
7. The sum of the lengths of any two sides of a triangle is greater than the length of the third side. (The Triangle Inequality)
8. You can use the SAS Inequality and SSS Inequality Theorems to compare the lengths of sides and measures of angles in two triangles.

Chapter Review

Complete each statement by writing <, =, or >.

1. $m\angle 1$ __?__ $m\angle 5$ 6–1
2. $m\angle 1$ __?__ $m\angle 2$
3. $m\angle 3$ __?__ $m\angle 4$
4. $m\angle 5$ __?__ $m\angle 2$
5. If $a > b$, $c < b$, and $d = c$, then a __?__ d.



Given: All registered voters must be at least 18 years old.
What, if anything, can you conclude from each additional statement?

6. Eric is 19 years old. 6–2
7. Bonnie is not registered to vote.
8. Will is 15 years old.
9. Barbara is a registered voter.

10. Write the letters (a)–(d) in an order that completes an indirect proof of the statement: If $n^2 + 6 = 32$, then $n \neq 5$. 6–3
(a) But this contradicts the fact that $n^2 + 6 = 32$.
(b) Our temporary assumption must be false, and it follows that $n \neq 5$.
(c) Assume temporarily that $n = 5$.
(d) Then $n^2 + 6 = 31$.

11. In $\triangle TOP$, if $OT > OP$, then $m\angle P >$ _?_. 6–4

12. In $\triangle RED$, if $m\angle D < m\angle E$, then $RD >$ _?_.

13. Points X and Y are in plane M. If $\overline{PX} \perp$ plane M, then PX _?_ PY.

14. Two sides of a triangle have lengths 6 and 8. The length of the third side must be greater than _?_ and less than _?_.

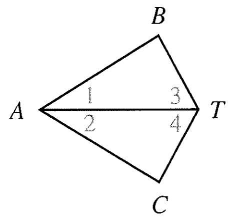
Complete each statement by writing <, =, or >.

15. If $\overline{AB} \cong \overline{AC}$ and $m\angle 1 > m\angle 2$, then BT _?_ CT. 6–5

16. If $\overline{TB} \cong \overline{TC}$ and $AB < AC$, then $m\angle 3$ _?_ $m\angle 4$.

17. If $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, then AB _?_ AC.

18. If $\overline{TB} \cong \overline{TC}$ and $m\angle 3 > m\angle 4$, then AB _?_ AC.



Chapter Test

Complete each statement by writing <, =, or >.

1. If $x > y$ and $y = z$, then x _?_ z.

2. If $a > b$, and $c < b$, then c _?_ a.

3. If $s = t + 4$, then s _?_ t.

4. If $e + 5 = f + 4$, then e _?_ f.

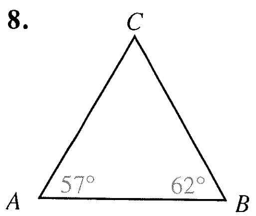
5. Write (a) the inverse and (b) the contrapositive of
"If point P is on $\overline{AB}$, then $AB > AP$."

6. Pair each statement below with the given statement above and tell what conclusion, *if any*, must follow.
a. P is not on $\overline{AB}$. **b.** P is on $\overline{AB}$. **c.** $AB \leq AP$ **d.** $AB > AP$

7. If the lengths of the sides of a triangle are x, 15, and 21, then x must be greater than _?_ and less than _?_.

In Exercises 8–10 the diagrams are not drawn to scale. If each diagram were drawn accurately, which segment shown would be the shortest?

8.



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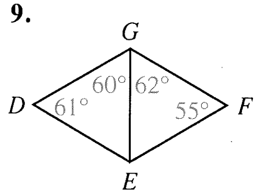

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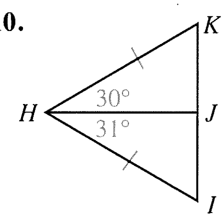
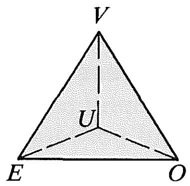


11. If $VE > VO$, then $m\angle$ _?_ $> m\angle$ _?_.

12. If $m\angle UEO > m\angle UOE$, then _?_ $>$ _?_.

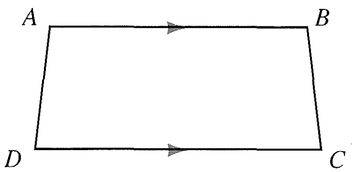
13. If $\overline{VE} \cong \overline{VO}$ and $m\angle UVE > m\angle UVO$, then _?_ $>$ _?_.

14. If $m\angle EVU = 60$, $\overline{OE} \cong \overline{OU}$, and $m\angle VOE > m\angle VOU$, then the largest angle of $\triangle UVE$ is $\angle$ _?_.

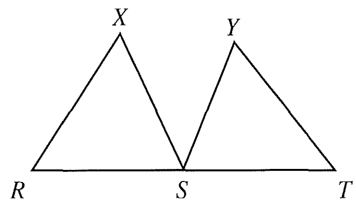


Exs. 11–14

15. Write an indirect proof.
Given: Trap. $ABCD$ with $\overline{AB} \parallel \overline{DC}$
Prove: $\angle C$ and $\angle D$ are not both right angles.



16. Given: $XS > YS$; $\overline{RX} \cong \overline{TY}$;
S is the midpoint of $\overline{RT}$.
Prove: $m\angle R > m\angle T$



Algebra Review: *Fractions*

Simplify the following fractions.

Example **a.** $\frac{8w}{2}$ **b.** $\frac{5t - 10}{15}$ **c.** $\frac{x + 6}{36 - x^2}$

Solution **a.** $4w$ **b.** $\frac{5(t - 2)}{15} = \frac{t - 2}{3}$ **c.** $\frac{x + 6}{(6 - x)(6 + x)} = \frac{1}{6 - x}$

1. $\frac{14}{70}$ **2.** $\frac{75}{15}$ **3.** $\frac{18a}{36}$ **4.** $\frac{3x}{x}$

5. $\frac{x}{3x}$ **6.** $\frac{5bc}{10b^2}$ **7.** $\frac{-8y^3}{2y}$ **8.** $\frac{-18r^3t}{12rt}$

9. $\frac{3ab^2}{6bc}$ **10.** $\frac{6a + 12}{6}$ **11.** $\frac{9x - 6y}{3}$ **12.** $\frac{33ab - 22b}{11b}$

13. $\frac{x + 2}{3x + 6}$ **14.** $\frac{2c - 2d}{2c + 2d}$ **15.** $\frac{t^2 - 1}{t - 1}$ **16.** $\frac{5a + 5b}{a^2 - b^2}$

17. $\frac{b^2 - 25}{b^2 - 12b + 35}$ **18.** $\frac{a^2 + 8a + 16}{a^2 - 16}$ **19.** $\frac{3x^2 - 6x - 24}{3x^2 + 2x - 8}$

Preparing for College Entrance Exams

Strategy for Success

You may find it helpful to sketch figures or do calculations in your test booklet. Be careful not to make extra marks on your answer sheet.

Indicate the best answer by writing the appropriate letter.

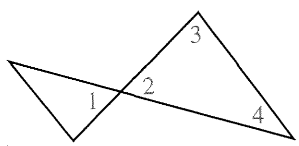
1. The diagonals of quadrilateral $MNOP$ intersect at X. Which statement guarantees that $MNOP$ is a rectangle?
(A) $MX = NX = OX = PX$ **(B)** $\angle PMN \cong \angle MNO \cong \angle NOP$
(C) $MO = NP$ **(D)** $\overline{MO} \perp \overline{NP}$ **(E)** $\overline{MN} \perp \overline{MP}$

2. Which statement does *not* guarantee that quadrilateral $WXYZ$ is a parallelogram?
(A) $\overline{WX} \cong \overline{YZ}$; $\overline{XY} \parallel \overline{WZ}$ **(B)** $\angle W \cong \angle Y$; $\angle X \cong \angle Z$
(C) $\overline{WX} \cong \overline{YZ}$; $\overline{XY} \cong \overline{WZ}$ **(D)** $\overline{XY} \parallel \overline{WZ}$; $\overline{WX} \parallel \overline{ZY}$
(E) $\overline{XY} \cong \overline{WZ}$; $\overline{XY} \parallel \overline{WZ}$

3. In $\triangle ABC$, if $AB = BC$ and $AC > BC$, then:
(A) $AB < AC - BC$ **(B)** $m\angle B > m\angle C$ **(C)** $m\angle B < m\angle A$
(D) $m\angle B = 60$ **(E)** $m\angle B = m\angle A$

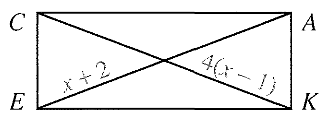
4. Which statement is not always true for every rhombus $ABCD$?
(A) $AB = BC$ **(B)** $AC = BD$ **(C)** $\angle B \cong \angle D$
(D) $\overline{AC} \perp \overline{BD}$ **(E)** $\angle ABD \cong \angle CBD$

5. Given: $m\angle 3 > m\angle 4$
Compare: $x = m\angle 1 + m\angle 4$ and
$y = m\angle 2 + m\angle 3$
(A) $x > y$ **(B)** $y > x$ **(C)** $x = y$
(D) No comparison possible with information given



6. Given: Trapezoid $LMNO$; $\overleftrightarrow{MN} \parallel \overleftrightarrow{LO}$; $\overline{LO}$ is twice as long as $\overline{MN}$. How long is the median of the trapezoid?
(A) $\frac{4}{3}LO$ **(B)** $\frac{3}{2}LO$ **(C)** $\frac{2}{3}MN$ **(D)** $\frac{3}{4}MN$ **(E)** $\frac{3}{2}MN$

7. Quad. $CAKE$ is a rectangle. Find CK.
(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 8



8. Which of the following statement(s) are true?
(I) If $a > b$, then $ax > bx$ for all numbers x.
(II) If $ax > bx$ for some number x, then $a > b$.
(III) If $a > b$, then for some number x, $ax < bx$.
(A) I only **(B)** II only **(C)** III only **(D)** all of the above
(E) none of the above

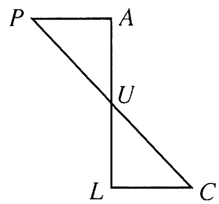
Cumulative Review: Chapters 1–6

A

1. An angle and its complement have the measures $x + 38$ and $2x - 5$. Find the measure of the angle.

2. Find the sum of the measures of the interior angles of a pentagon.

3. Can the given information be used to prove the triangles congruent? If so, which congruence postulate or theorem would you use?



a. Given: $\overline{PC}$ and $\overline{AL}$ bisect each other.

b. Given: $\angle P \cong \angle C$; U is the midpoint of $\overline{PC}$.

c. Given: $\overline{PA} \parallel \overline{LC}$

d. Given: $\overline{PA} \perp \overline{AL}$; $\overline{LC} \perp \overline{AL}$; $\overline{PU} \cong \overline{UC}$

4. Tell whether the statement is *always*, *sometimes*, or *never* true for a parallelogram $ABCD$ with diagonals that intersect at P.

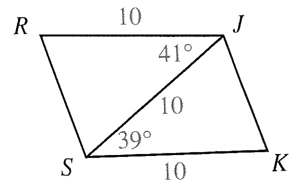
a. $AB = BC$ **b.** $\overline{AC} \perp \overline{BD}$ **c.** $\angle A$ and $\angle B$ are complementary $\angle$s.

d. $\angle ADB \cong \angle CBD$ **e.** $\overline{AP} \cong \overline{PC}$ **f.** $\triangle ABC \cong \triangle CDA$

5. In $\triangle XYZ$, $m\angle X = 64$ and $m\angle Y = 54$. Name **(a)** the longest and **(b)** the shortest side of $\triangle XYZ$.

6. a. Which segment is longer: $\overline{RS}$ or $\overline{JK}$?

b. Name the theorem that supports your answer.



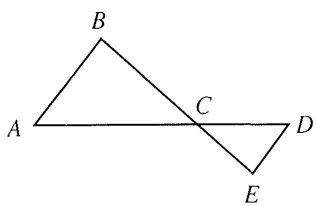
B

7. The difference between the measures of two supplementary angles is 38. Find the measure of each angle.

8. The lengths of the sides of a triangle are z, $z + 3$, and $z + 6$. What can you conclude about the value of z?

9. Write an indirect proof of the following statement: If $PQRS$ is a quadrilateral, then $\angle Q$, $\angle R$, and $\angle S$ are not all 120°.

10. Given: $m\angle B > m\angle A$;
$m\angle E > m\angle D$
Prove: $AD > BE$



11. Given: $\overline{DC} \parallel \overline{AB}$; $\overline{CE} \perp \overline{AB}$; $\overline{AF} \perp \overline{AB}$
Prove: $AECF$ is a rectangle.